# Derivatives Series 1, Power Rule for Integers

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## 1 Introduction

Hello, this will be the first part of my math proof series. I do a lot of math proofs on my own, and I figured I should probably have a place to keep them, also I could use practice with LaTeXwrite ups.

I decided to start with my power rule proof as it is one of the more simple ones, but it is still very fun. Also, it is a calculus proof so that's also fun.

The rule for this is that I am not allowed any outside help and can only use other things I have proved in my proofs. Still there are many spoilers so I try to avoid them, so far the only major spoilers I have had are knowing that the rule exists.

## 2 Goal

find

$$\frac{d}{dx}[x^a]$$

Where the constant a = Any Real Integer

## 3 Proof for a = Positive Integer

1. Definition of the Derivative

$$\frac{d}{dx}[x^a] = \lim_{dx \to 0} \frac{(x + dx)^a - x^a}{dx}$$

2. Binomial expansion (since a binomial expansion can only be done with a positive integer exponent a second section for the case of a negative integer exponent is necessary)

$$\lim_{dx \to 0} \frac{(x^a dx^0 + ax^{a-1}dx^1 + bx^{a-2}dx^2 ... + x^0 dx^a) - x^a}{dx}$$

It should be noted that this series for a binomial expansion to the ath power continues with increasing powers of dx as shown, and also that the coefficient of the 2nd term is "a" as it is the amount of combinations of n-1 x(s) and 1 dx that can be made when multiplying binomials. Also the coefficients for the other terms besides the 1st and 2nd are irrelevant, so the "b" coefficient is simply to show that here is a coefficient there, but still its value does not matter.

As an example all combinations for a = 4 are as followes:
x x x dx — 1
x x dx x — 2
x dx x x — 3
dx x x x — 4

3. Cancelling out like terms

$$\lim_{dx \to 0} \frac{ax^{a-1}dx^1 + bx^{a-2}dx^2 ... + x^0 dx^a}{dx}$$

4. Dividing by dx

$$\lim_{dx \to 0} ax^{a-1}dx^0 + bx^{a-2}dx^1 ... + x^0 dx^{a-1}$$

5. Subbing in for the limit and evaluating

$$ax^{a-1}0^0 + bx^{a-2}0^1 ... + x^0 0^{a-1}$$

Note: $0^0 = 1$, $0^n = 0\ \forall n > 0$

$$ax^{a-1}$$

Like stated previous the other coefficients of terms passed the 2nd term, like "b", are irrelevant as they are multiplied by 0 when the limit is evaluated.

6. Final answer

$$\frac{d}{dx}[x^a] = \lim_{dx \to 0} \frac{(x+dx)^a - x^a}{dx} = ax^{a-1}$$

$$\frac{d}{dx}[x^a] = ax^{a-1}$$

# 4 Proof for a = Negative Integer

1. a is a Negative Integer, Thus

$$a = -b$$

Where "b" is a Positive integer

2. Definition of the Derivative

$$\frac{d}{dx}[x^a] = \frac{d}{dx}[x^{-b}] = \lim_{dx\to 0} \frac{(x+dx)^{-b} - x^{-b}}{dx}$$

3. Re-write to make all exponents positive

$$\lim_{dx\to 0} \frac{\frac{x^b}{x^b(x+dx)^b} - \frac{(x+dx)^b}{x^b(x+dx)^b}}{dx}$$

$$\lim_{dx\to 0} \frac{\frac{x^b-(x+dx)^b}{dx}}{x^b(x+dx)^b}$$

$$\lim_{dx\to 0} -\frac{\frac{(x+dx)^b-x^b}{dx}}{x^b(x+dx)^b}$$

4. Evaluating the top portion (refer to "Proof for a = Positive Integer" lines 1-4)

$$\lim_{dx\to 0} -\frac{bx^{b-1}dx^0 + cx^{b-2}dx^1 ... + x^0dx^{b-1}}{x^b(x+dx)^b}$$

5. Evaluating the bottom portion (binomial expansion)

$$\lim_{dx\to 0} -\frac{bx^{b-1}dx^0 + cx^{b-2}dx^1 ... + x^0dx^{b-1}}{x^b(x^bdx^0 + bx^{b-1}dx^1 ... + x^0dx^b)}$$

$$\lim_{dx\to 0} -\frac{bx^{b-1}dx^0 + cx^{b-2}dx^1 ... + x^0dx^{b-1}}{x^{2b}dx^0 + bx^{2b-1}dx^1 ... + x^bdx^b}$$

6. Evaluating limit and simplifying

$$-\frac{bx^{b-1}0^0 + cx^{b-2}0^1 ... + x^0 0^{b-1}}{x^{2b}0^0 + bx^{2b-1}0^1 ... + x^b 0^b}$$

$$-\frac{bx^{b-1}1}{x^{2b}1}$$

$$-bx^{b-1-2b}$$

$$-bx^{-b-1}$$

7. Subbing a back in for -b

$$ax^{a-1}$$

8. Final answer

   Where a = Negative Integer

$$\frac{d}{dx}[x^a] = \frac{d}{dx}[x^{-b}] = \lim_{dx\to 0} \frac{(x+dx)^{-b} - x^{-b}}{dx} = -bx^{-b-1} = ax^{a-1}$$

$$\frac{d}{dx}[x^a] = ax^{a-1}$$