Derivatives Series 2, Multiplication Rule

Patrick Dugan

Originally Made: Sometime early in the summer before my sophomore year
Re-Written in LaTex : February 2021

1 Introduction

The rule for this is that I am not allowed any outside help and can only use other things I have proved in my proofs. Still there are many spoilers so I try to avoid them, so far the only major spoilers I have had are knowing that the rule exists.

Multiplication Rule is very helpful, so I want to be able to use it for future proofs.

2 Notation

$$df = f(x + dx) - f(x)$$

$$\lim_{dx \to 0} \frac{df}{dx} = f'(x)$$

(This is just the definition of a derivative)

3 Goal

For any functions f(x) and g(x) Express:

$$\frac{d}{dx}[f(x)g(x)]$$

As a combination of $f(x), f'(x), g(x)$, and $g'(x)$

4 Lemma 1

Goal:

$$\frac{d}{dx}[f(x)g(x)] = \lim_{dx \to 0} \frac{df[g(x + dx) + g(x)] + f(x)g(x + dx) + -f(x + dx)g(x)}{dx}$$

1. Definition of the derivative

$$\frac{d}{dx}[f(x)g(x)] = \lim_{dx \to 0} \frac{f(x + dx)g(x + dx) - f(x)g(x)}{dx}$$

2. Insanely clever factoring

$$\lim_{dx \to 0} \frac{[f(x + dx) - f(x)][g(x + dx) + g(x)] + f(x)g(x + dx) + -f(x + dx)g(x)}{dx}$$

3. Notation ($df = f(x + dx) - f(x)$)

$$\lim_{dx \to 0} \frac{df[g(x + dx) + g(x)] + f(x)g(x + dx) + -f(x + dx)g(x)}{dx}$$

5 Lemma 2

Goal:

$$\lim_{dx\to 0} \frac{df[g(x+dx)+g(x)] + f(x)g(x+dx) + -f(x+dx)g(x)}{dx} = \lim_{dx\to 0} \frac{dg[f(x+dx)+f(x)] + g(x)f(x+dx) + -g(x+dx)f(x)}{dx}$$

1. Duhhh
$$\frac{d}{dx}[f(x)g(x)] = \frac{d}{dx}[f(x)g(x)]$$
2. Associative property
$$\frac{d}{dx}[f(x)g(x)] = \frac{d}{dx}[g(x)f(x)]$$
3. Lemma 1 (on both sides)
$$\lim_{dx\to 0} \frac{df[g(x+dx)+g(x)] + f(x)g(x+dx) + -f(x+dx)g(x)}{dx} = \lim_{dx\to 0} \frac{dg[f(x+dx)+f(x)] + g(x)f(x+dx) + -g(x+dx)f(x)}{dx}$$

6 Lemma 3

Goal:

$$\lim_{dx\to 0} 2(\frac{df[g(x+dx)+g(x)] + f(x)g(x+dx) + -f(x+dx)g(x)}{dx}) = \lim_{dx\to 0}[g(x+dx)+g(x)]\frac{df}{dx} + [f(x+dx)+f(x)]\frac{dg}{dx}$$

1. Multiplication
$$\lim_{dx\to 0}(\frac{df[g(x+dx)+g(x)] + f(x)g(x+dx) + -f(x+dx)g(x)}{dx}) + (\frac{df[g(x+dx)+g(x)] + f(x)g(x+dx) + -f(x+dx)g(x)}{dx})$$
2. Lemma 2
$$\lim_{dx\to 0}(\frac{df[g(x+dx)+g(x)] + f(x)g(x+dx) + -f(x+dx)g(x)}{dx}) + (\frac{dg[f(x+dx)+f(x)] + g(x)f(x+dx) + -g(x+dx)f(x)}{dx})$$
3. Adding and reordering for clarity
$$\lim_{dx\to 0} \frac{df[g(x+dx)+g(x)] + dg[f(x+dx)+f(x)] + f(x)g(x+dx) + -g(x+dx)f(x) + g(x)f(x+dx) + -f(x+dx)g(x)}{dx}$$
4. Canceling out like terms (yeah I know whoah moment)
$$\lim_{dx\to 0} \frac{df[g(x+dx)+g(x)] + dg[f(x+dx)+f(x)]}{dx}$$
5. Rearranging for clarity
$$\lim_{dx\to 0}[g(x+dx)+g(x)]\frac{df}{dx} + [f(x+dx)+f(x)]\frac{dg}{dx}$$

7 Proof

Goal:

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + g'(x)f(x)$$

1. Lemma 1
$$\frac{d}{dx}[f(x)g(x)] = \lim_{dx\to 0} \frac{df[g(x+dx)+g(x)] + f(x)g(x+dx) + -f(x+dx)g(x)}{dx}$$
2. Multiply both sides by 2
$$2\frac{d}{dx}[f(x)g(x)] = \lim_{dx\to 0} 2\frac{df[g(x+dx)+g(x)] + f(x)g(x+dx) + -f(x+dx)g(x)}{dx}$$

3. Lemma 3
$$2\frac{d}{dx}[f(x)g(x)] = \lim_{dx\to 0}[g(x+dx)+g(x)]\frac{df}{dx} + [f(x+dx)+f(x)]\frac{dg}{dx}$$

4. Solving limit ($\lim_{dx\to 0}\frac{df}{dx} = f'(x)$)
$$2\frac{d}{dx}[f(x)g(x)] = [g(x)+g(x)]f'(x) + [f(x)+f(x)]g'(x)$$

5. Combine like terms
$$2\frac{d}{dx}[f(x)g(x)] = 2g(x)f'(x) + 2f(x)g'(x)$$

6. Divide both sides by 2
$$\frac{d}{dx}[f(x)g(x)] = g(x)f'(x) + f(x)g'(x)$$