Derivatives 4, E!!!! and Derivatives of Exponentials

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I have had the bits and pieces of this derivation for a long time but at the beginning of sophomore year the full structure of this derivation popped into my head.
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(trigger warning: these derivations are going to be some of the least rigorous things you have ever seen), Also I am doing this from scratch off the cuff so my notation for some of this earlier stuff is going to be pretty weird.

1 Introduction

The rule for this is that I am not allowed any outside help and can only use other things I have proved in my derivations. Still there are many spoilers so I try to avoid them, so far the only major spoilers I have had are knowing that the rule exists.

love this derivation, yeah its e time.

2 Goal

figure out

$$\frac{d}{dx}[a^x]$$

where a is a constant, non negative

3 Lemma 1 $\frac{d}{dx}[a^x] = a^x(\lim_{dx\to 0} \frac{a^{dx}-1}{dx})$

a = non negative constant

1. Definition of the derivative
$$\frac{d}{dx}[a^x] = (\lim_{dx\to 0} \frac{a^{x+dx} - a^x}{dx})$$

2. Factor out a^x, this is already interesting because it shows that the derivative of an exponential is itself times a constant
$$\frac{d}{dx}[a^x] = a^x(\lim_{dx\to 0} \frac{a^{dx} - 1}{dx})$$

4 Lemma 2 $\frac{d}{dx}[e^x] = e^x$

We are pretty much defining e, and $\frac{d}{dx}[e^x]$

1. Lemma 1 a is any non negative constant
$$\frac{d}{dx}[a^x] = a^x(\lim_{dx\to 0}\frac{a^{dx}-1}{dx})$$
2. Define a constant e (wow wonder what this could be) such that $\lim_{dx\to 0}\frac{e^{dx}-1}{dx} = 1$
$$\frac{d}{dx}[e^x] = e^x(\lim_{dx\to 0}\frac{e^{dx}-1}{dx}) = e^x(1)$$
3. Thus
$$\frac{d}{dx}[e^x] = e^x$$

5 Derivation

Goal:
$$\frac{d}{dx}[a^x] = a^x(ln(a))$$

1. Reorganising, lets say for instance that the log of base e is ln(x) (shocking)
$$\frac{d}{dx}[a^x] = \frac{d}{dx}[e^{xln(a)}]$$
2. Chain Rule (Refer to the chain rule derivation)
$$\frac{d}{dx}[a^x] = \frac{d}{dx}[x(ln(a))]\frac{d}{dx}[e^x]_{x=xln(a)} = (ln(a))\frac{d}{dx}[e^x]_{x=xln(a)}$$
3. Lemma 2
$$\frac{d}{dx}[a^x] = e^{xln(a)}(ln(a))$$
4. Definition of the log $e^{ln(x)} = x$ This now tells us what that cryptic limit was
$$\frac{d}{dx}[a^x] = a^x(ln(a))$$

6 Fun side notes with E

6.1 Calculating e

There are two main ways I have found. First, due to e^x being its own derivative the Taylor series at 0 will have derivatives that are all the same at zero (aka 1). when this Taylor series is evaluated at 1 (e^1) will calculate e. The series is $\sum_{n=0}^{\infty}\frac{1}{n!} = e$.
But this sum isn't the method I am going to show the one I will show is the limit

1. Definition of e that we established at the beginning of the derivation

$$1 = (\lim_{dx \to 0} \frac{e^{dx} - 1}{dx})$$

2. Algebra (now is it rigorous to break up an indeterminate for like this... idk, but I will)

$$\lim_{dx \to 0} dx = \lim_{dx \to 0} e^{dx} - 1$$

3. Algebra

$$\lim_{dx \to 0} 1 + dx = \lim_{dx \to 0} e^{dx}$$

4. Algebra (last time)

$$\lim_{dx \to 0} (1 + dx)^{\frac{1}{dx}} = \lim_{dx \to 0} e$$

5. No limit variables on the right side so limit can be removed

$$\lim_{dx \to 0} (1 + dx)^{\frac{1}{dx}} = e$$

7 Derivation of derivative of logs

Goal:

$$\frac{d}{dx}[log_a(x)] = \frac{1}{xln(x)}$$

7.1 lemma 1 $\frac{d}{dx}[ln(x)] = \frac{1}{x}$

1. Def of ln(x)

$$e^{ln(x)} = x$$

2. take derivative of both sides

$$\frac{d}{dx}[ln(x)]e^{ln(x)} = 1$$

3. divide both sides

$$\frac{d}{dx}[ln(x)] = \frac{1}{e^{ln(x)}}$$

4. definition of a limit

$$\frac{d}{dx}[ln(x)] = \frac{1}{x}$$

7.2 $\frac{d}{dx}[log_a(x)] = \frac{1}{xln(x)}$

1. log change of base rule $\frac{d}{dx}[log_a(x)] = \frac{d}{dx}[\frac{ln(x)}{ln(a)}]$
2. constants can be taken out of the limit that is the derivative $\frac{1}{ln(a)} \frac{d}{dx}[ln(x)]$
3. derivative of ln(x) $\frac{1}{xln(a)}$

8 Random cool side notes

8.1 cool stuff with ln and logs in general

Thought this was cool and it eventually helped me prove a integral later on (there were other ways to do it that I end up finding but I was able to use this to by pass those things for that specific integral, but back on topic)

1. They are identical
$$\frac{d}{dx}[a^x] = \frac{d}{dx}[a^x]$$
2. The derivative rule we made
$$a^x(ln(a)) = a^x(\lim_{dx \to 0} \frac{a^{dx} - 1}{dx})$$
3. Algebra
$$ln(a) = \lim_{dx \to 0} \frac{a^{dx} - 1}{dx}$$
This is pretty cool on its own, it sort of gives insight to the graphical similarity to fractional polynomials (roots) and logarithmic functions. Still, you can go a little further
4. Division and using the previous equality (I don't feel like making the above a Lemma you can figure it out)
$$\frac{ln(a)}{ln(b)} = \frac{\lim_{dx \to 0} \frac{a^{dx}-1}{dx}}{\lim_{dx \to 0} \frac{b^{dx}-1}{dx}}$$
5. Log change of base rule and simplifying fractions
$$\log_b a = \lim_{dx \to 0} \frac{a^{dx} - 1}{b^{dx} - 1}$$
Thought this limit was pretty cool so yeah thats it

8.2 Random super flex derivation for $\int_1^x \frac{1}{x}\, dx$

I have not gotten to writing the derivations for integrals yet, but this is a good spot for this side tangent. This is really pointless derivation and there are simple derivations for this with the same required theorems, more simple and understandable. Still this one is a super cool flex and kind of explains why logs look similar to roots. Also helps make you more comfortable with the fact that the power rule works on everything but x^{-1} as with limits it does. well enough talk.

1. Pretty straight forward limit
$$\frac{1}{x} = \lim_{dx \to 0} x^{dx-1}$$

2. Integrate
$$\int \frac{1}{x}\, dx = \int \lim_{dx \to 0} x^{dx-1}\, dx$$
3. Integral power Rule
$$\int \frac{1}{x}\, dx = [\frac{x^{dx}}{dx}]_1^x = \lim_{dx \to 0} \frac{x^{dx}}{dx} - \frac{1^{dx}}{dx}$$
4. Simplify
$$\int \frac{1}{x}\, dx = \lim_{dx \to 0} \frac{x^{dx} - 1}{dx}$$
5. That whole limit we saw a while ago
$$\int \frac{1}{x}\, dx = ln(x)$$