$e \notin \mathbb{Q}$

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1 What are we doing

I like irrationality proofs and often try to make my own. After trying to get one for years, I finally thought of something

The Plan is simple. Use properties of alternating series to get a bound on $\frac{-1}{e}$, use that bound and the assumption that e is rational to show that a non-integer is in the integers. Showing that the assumption is wrong

2 Alternating series

Alternating Series Givens

$$a_{n+1} \geq a_n$$

$$\lim_{n\to\infty} a_n = 0$$

$$a_n > 0$$

$$S_k = \sum_{n=1}^{k} (-1)^{n+1} a_n$$

Getting the bound

$$S_{2k+1} - a_{2k+2} = S_{2k+2}$$

1. -

$$S_{2k+1} > S_{2k+2}$$

Also

$$S_{2k} + (a_{2k+1} - a_{2k+2}) = S_{2k+2}$$

Note: $a_{2k+1} - a_{2k+2} > 0$ 2. -

$$S_{2k} < S_{2k+2}$$

Also

$$S_{2k+1} + (-a_{2k+2} + a_{2k+3}) = S_{2k+3}$$

3. -

$$S_{2k+1} > S_{2k+3}$$

Also

$$S_{2k+2} + a_{2k+3} = S_{2k+3}$$

4. -

$$S_{2k+2} < S_{2k+3}$$

Finally $S_{2k} < S_{2k+2} < S_{2k+1}$ and $S_{2k+2} < S_{2k+3} < S_{2k+1}$

3 S_g between S_k and S_{k+1}

Note: for a > b $S_{2b} < S_{2a}$ and $S_{2a+1} < S_{2b+1}$

for g > k + 1 (depending on if g is even or odd)

$$S_{g+1} < S_{g+2} < S_g \text{ or } S_g < S_{g+2} < S_{g+1}$$

k will be even (if not we could just swap ineq)

$$S_k < S_{g+1} < S_{g+2} < S_g < S_{k+1} \text{ or } S_k < S_g < S_{g+2} < S_{g+1} < S_{k+1}$$

Thus

$$\text{for even k } S_k < S_g < S_{k+1} \text{ and for odd k } S_{k+1} < S_g < S_k$$

4 $S_g \neq S_k + ca_{k+1}$

Ok aiming for this

$$g > k + 1 > 1 \text{ and } c \in \mathbb{Z}, S_g \neq S_k + ca_{k+1}$$

- when $c = 0$

$$\text{for even k } S_k < S_g \text{ and for odd k } S_g < S_k$$

$$\text{for even k } S_k \neq S_g \text{ and for odd k } S_g \neq S_k$$

- when $c > 0$

$$\text{for even k } S_g < S_{k+1} \text{ and for odd k } S_g < S_k$$

$$\text{for even k } S_g < S_k + a_{k+1} \text{ and for odd k } S_g < S_k$$

$$\text{for even k } S_g < S_k + ca_{k+1} \text{ and for odd k } S_g < S_k + ca_{k+1}$$

$$\text{for even k } S_g \neq S_k + ca_{k+1} \text{ and for odd k } S_g \neq S_k + ca_{k+1}$$

- when $c < 0$ (note: $0 > ca_k$)

$$\text{for even k } S_g > S_{k+1} \text{ and for odd k } S_g > S_k$$

$$\text{for even k } S_g > S_k + a_{k+1} \text{ and for odd k } S_g > S_k$$

$$\text{for even k } S_g > S_k + ca_{k+1} \text{ and for odd k } S_g > S_k + ca_{k+1}$$

$$\text{for even k } S_g \neq S_k + ca_{k+1} \text{ and for odd k } S_g \neq S_k + ca_{k+1}$$

5 $\frac{1}{a_k}\sum_{n=k+1}^{g}(-1)^{n+1}a_n \notin \mathbb{Z}$ $(g > k+1)$

Ok this is quick

$$S_g \neq S_k + ca_{k+1}$$

$$\frac{1}{a_{k+1}}(S_g - S_k) \neq c$$

(note c is any $\mathbb{Z}$)

$$\frac{1}{a_{k+1}}(S_g - S_k) \notin \mathbb{Z}$$

$$\frac{1}{a_k}\sum_{n=k+1}^{g}(-1)^{n+1}a_n \notin \mathbb{Z}$$

6 $e^{\frac{1}{\mathbb{Z}}} \notin \mathbb{Q}$

assume $e^{\frac{-1}{\mathbb{Z}}}$ is rational $a, b, c \in \mathbb{Z}$

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n!c^n} = \frac{a}{b}$$

$$(b!c^b)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!c^n} = \frac{b!c^b a}{b}$$

$$(b!c^b)\sum_{n=0}^{b}\frac{(-1)^n}{n!c^n} + (b!c^b)\sum_{n=b+1}^{\infty}\frac{(-1)^n}{n!c^n} = (b-1)!c^b a$$

$$(b!c^b)\sum_{n=b+1}^{\infty}\frac{(-1)^n}{n!c^n} = (b-1)!c^b a - \sum_{n=0}^{b}\frac{(-1)^n b!c^{b-n}}{n!}$$

$$(b!c^b)\sum_{n=b+1}^{\infty}\frac{(-1)^n}{n!c^n} = \mathbb{Z}$$

contradiction by section 5 => assumption is wrong => $e^{\frac{-1}{\mathbb{Z}}} \notin \mathbb{Q}$

$$e^{\frac{1}{\mathbb{Z}}} \notin \mathbb{Q}$$

7 Bonus $\sin(\frac{1}{\mathbb{Z}}) \notin \mathbb{Q}$

assume $\sin(\frac{1}{\mathbb{Z}})$ is rational $a, b, c \in \mathbb{Z}$

$$\sin(\frac{1}{c}) = \frac{a}{b}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!c^{2n+1}}=\frac{a}{b}$$

$$((2b+1)!c^{2b+1})\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!c^{2n+1}}=\frac{(2b+1)!c^{2b+1}a}{b}$$

$$((2b+1)!c^{2b+1})\sum_{n=0}^{b}\frac{1}{(2n+1)!c^{2n+1}}+((2b+1)!c^{2b+1})\sum_{n=b+1}^{\infty}\frac{(-1)^n}{(2n+1)!c^{2n+1}}=\frac{(2b+1)!c^{2b+1}a}{b}$$

$$((2b+1)!c^{2b+1})\sum_{n=b+1}^{\infty}\frac{(-1)^n}{(2n+1)!c^{2n+1}}=\frac{(2b+1)!c^{2b+1}a}{b}-\sum_{n=0}^{b}\frac{(-1)^n(2b+1)!c^{2(b-n)}}{(2n+1)!}$$

$$((2b+1)!c^{2b+1})\sum_{n=b+1}^{\infty}\frac{(-1)^n}{(2n+1)!c^{2n+1}}=\mathbb{Z}$$

contradiction by section 5 => assumption is wrong => $\sin(\frac{1}{\mathbb{Z}})\notin\mathbb{Q}$