1. Some Results with Kernels

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1 Kernel Notation

$$\text{Where f is an endomorphism : } H \to H$$

$$K_H(f) = \{x : x \in H, f(x) = e_H\}$$

1.1 Subgroups

$$G \leq H$$

$$K_G(f) \leq K_H(f)$$

1.2 Factor Groups

If you think about endomorphism as factor groups of the kernel. Then considering this equation $f(a) = c$ The coset of the kernel $ker(f) * a$ comprises all solutions to $f(x) = c$. With this one can see that every coset of the kernel corresponds to one element in the image.

$$f(H) \cong H/K_H(f)$$

Thinking about this as we consider the Kernel of f on the image of g. Each value in the kernel of f corresponds to a value in the image of g which is a coset of $K_H(g)$

$$K_H(f \circ g)/K_H(g) \cong K_{g(H)}(f)$$

I understand this as the overlap between the kernel of f and the image of g

1.3 order

$$K_H(f \circ g)/K_H(g) \cong K_{g(H)}(f) \leq K_H(f)$$

$$K_H(f \circ g)/K_H(g) \leq K_H(f)$$

$$\frac{|K_H(f \circ g)|}{|K_H(g)|} \text{ divides } |K_H(f)|$$

finally

$$|K_H(f \circ g)| \text{ divides } |K_H(f)| * |K_H(g)|$$

This is a neat statement that we are going to use a lot.

2 Kernels of x^n

2.1 $K_H(x^{p_i})$

Say H is an abelian group and p_i is prime, p_i divides $|H|$

$$|H| = p_1^{m_1} p_2^{m_2} ...\ p_i^{m_i}$$

$$\text{For all } x \in K_H(x^{p_i}),\ x^{p_i} = 1$$

Since the order of x is p_i,

$$< x > \quad \cong C_{p_i}$$

.

$$K_H(x^{p_i}) \text{ is made up of disjoint } C_{p_i} \text{ groups}$$

By repeatedly making factor groups with sub group one can see

$$|K_H(x^{p_i})| = p_i^c$$

c is some integer (or seeing this as a direct product is probably clearer)

2.2 $K_H(x^{p_i^{m_i}})$

$$|H| = p_1^{m_1} p_2^{m_2} ...\ p_i^{m_i}$$

$$K_H(x^{p_i^{m_i}}) = K_H(x^{p_i} \circ x^{p_i} \circ ...\ x^{p_i})$$

Using the relationship from the 1.3 and 2.1

$$|K_H(x^{p_i^{m_i}})| \text{ divides } p_i^c * p_i^c * ...\ p_i^c$$

using the fact that p is prime

$$|K_H(x^{p_i^{m_i}})| = p_i^{k_i}$$

where k_i is some integer

2.3 Solving for k_i

$$K_H(x^{|H|}) = K_H(x^{p_1^{m_1}} \circ x^{p_2^{m_2}} \circ ...\ x^{p_i^{m_i}})$$

Using 1.3, 2.2 and that any element to the order of the group is identity

$$|H| \text{ divides } p_1^{k_1} * p_2^{k_2} * ...\ p_i^{k_i}$$

$$p_1^{m_1} p_2^{m_2} ...\ p_i^{m_i} \text{ divides } p_1^{k_1} * p_2^{k_2} * ...\ p_i^{k_i}$$

Using proprieties of primes this means

$$p_i^{m_i} \text{ divides } p_i^{k_i}$$

$$m_i \leq k_i$$

Also since $K_H(x^{p_i^{m_i}}) \leq H$

$$|K_H(x^{p_i^{m_i}})| \text{ divides } |H|$$

$$p_i^{k_i} \text{ divides } p_1^{m_1} p_2^{m_2} ... \ p_i^{m_i}$$

again using proprieties of primes

$$p_i^{k_i} \text{ divides } p_i^{m_i}$$

$$k_i \leq m_i$$

So finally!

$$m_i \leq k_i \leq m_i$$

$$k_i = m_i$$

So this means that for an abelian group H where $|H| = p_1^{m_1} p_2^{m_2} ... \ p_i^{m_i}$

$$|K_H(x^{p_i^{m_i}})| = p_i^{m_i}$$

3 Direct Products

This is pretty straight forward, but just want to make sure I am not assuming something without proof.
Say I have a group, and two disjoint subgroups of H, A and B

$$A, B \leq H$$

Considering the group (call it C) of elements that can be made by A and B

$$C = \{xy : x \in A, y \in B\}, C \leq H$$

Are all combinations of elements from A and B unique?

$$\text{assume not: } A_i * B_j = A_c * B_g$$

$$A_c^{-1} * A_i = B_g * B_j^{-1}$$

$$A_k = B_m \text{ contradiction as they are disjoint}$$

this means all elements in C can be represented by a unique combination of elements in A and B

So $C \cong A \times B$

3.1 Kernels are disjoint

Going back to those kernels of H

$$A, K_H(x^{p_a^{m_a}}) \leq H$$

where A is a direct product of kernels (not including $K_H(x^{p_a^{m_a}})$)

$$A \cong K_H(x^{p_1^{m_1}}) \times K_H(x^{p_2^{m_2}}) \times ...$$

We can show that these groups are disjoint. First assume they are not.

Say there exists y such that $y \in A$ and $y \in K_H(x^{p_a^{m_a}})$

An element's order must divide the order of it's group (can make a cyclic subgroup with that order) first

So $|y|$ must divide $|K_H(x^{p_a^{m_a}})|$

$|y| = p_a^c$ for some c

Also

$|y|$ divides $|A|$

Since $|A| = |K_H(x^{p_1^{m_1}})| * |K_H(x^{p_2^{m_2}})| * ...$

p_a^c divides $p_1^{m_1} p_2^{m_2} ...$

Since the two numbers are co-prime this is a contradiction

$K_H(x^{p_a^{m_a}})$ and $K_H(x^{p_1^{m_1}}) \times K_H(x^{p_2^{m_2}}) \times ...$ are disjoint

3.2 Breaking up H

Reminder: $|H| = p_1^{m_1} p_2^{m_2} ... \, p_i^{m_i}$ Now that we can form a direct product, lets call the Resulting group C.

$$C \cong K_H(x^{p_1^{m_1}}) \times K_H(x^{p_2^{m_2}}) \times ... \times K_H(x^{p_i^{m_i}})$$

$$|C| = p_1^{m_1} p_2^{m_2} ... \, p_i^{m_i} = |H|$$

So we have this group C where we know: $C \leq H$ and $|C| = |H|$ This means $C \cong H$, so finally

For any abelian H, $|H| = p_1^{m_1} p_2^{m_2} ... \, p_i^{m_i}$

$$H \cong K_H(x^{p_1^{m_1}}) \times K_H(x^{p_2^{m_2}}) \times ... \times K_H(x^{p_i^{m_i}})$$

4 Cyclic groups and more generators

This is a little overkill, but thought it was neat use of some of the results
Say we have abelian group G and $|G| = p_1 * p_2 * ... * p_i$

$$|K_G(x^{p_i})| = p_i$$

$$K_G(x^{p_i}) \cong C_{p_i}$$

$$G \cong K_G(x^{p_1^{m_1}}) \times K_G(x^{p_2^{m_2}}) \times ... \times K_G(x^{p_i^{m_i}})$$

$$G \cong C_{p_1} \times C_{p_2} \times ... \times C_{p_i}$$

G is a direct product of co prime cyclic groups and thus cyclic

$$G \cong C_{|G|}$$